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BONDED DISSIMILAR STRIPS WITH A CRACK PERPENDICULAR TO THE FUNCTIONALLY GRADED INTERFACE

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Abstract—The mode I crack problem for two dissimilar infinite strips bonded through a functionally graded interfacial zone is investigated within the framework of plane elasticity. The graded interfacial zone is treated as a nonhomogeneous continuum, having the continuously varying elastic modulus between the dissimilar homogeneous strips. A crack is assumed to be embedded in one of the strips perpendicular to the nominal interface. With the aid of the stiffness matrix approach, a set of homogeneous conditions relevant to the proposed crack problem is readily satisfied. Subsequent application of the mixed conditions on the cracked plane then leads to a singular integral equation of the first kind which is solved numerically. In consequence, the variations of stress intensity factors are provided as functions of geometric and material parameters of the bonded structure. The effect of the material nonhomogeneity in the interfacial zone is further addressed by measuring the degree of correspondence with the results that are obtained based on the use of the homogenized interfacial elastic property. Copyright © 1996 Elsevier Science Ltd.

1. INTRODUCTION

Over the past decades, a vast amount of research has been directed toward finding the solutions to crack problems for bonded materials, motivated by the potential technological advances that can be achieved by utilizing such multiphase media in many engineering situations. The classical solutions to this category of mixed boundary values problems can be largely attributed to England (1965) and Rice and Sih (1965) for the case of a crack located along the interface between bonded dissimilar materials, and to Zak and Williams (1963), Cook and Erdogan (1972), and Gupta (1973) when the crack is located perpendicular to the bimaterial interface. A common feature of these previous investigations is the assumption of the discrete nature of piecewise homogeneous structures with ideal interfaces, across which the elastic moduli are discontinuous. Because of such a drawback in the interface modeling, the pathological aspects of the oscillatory or the nonsquare-root singularity were exhibited, respectively, depending upon whether the crack-tip lies along or terminates at the interface.

Somewhat later, a suggestion was made by Atkinson (1977) as to how the ideal interface model that has been in use can be changed in order to remove the above unusual inconsistencies present in the analysis of crack problems. A nonhomogeneous interlayer was introduced at the location of the interface between the two different elastic media in which the shear modulus varies continuously. As a result, it was shown that the standard square-root type crack-tip singularity is maintained for both the crack geometries provided the spatial variation of the elastic modulus is continuous near and at the crack tips, despite the discontinuity in the derivative of the modulus.

It is worthwhile to mention at this point that the high resolution line scans by the electron microprobe indicated, in most diffusion bonded materials, the formation of a transitional phase with steeply varying physical and chemical properties between the two dinstinctively dissimilar substrates (Wagner *et al.*, 1995). Furthermore, in the diffusion bonded materials, the interfacial region is deliberately introduced and graded with the aim of producing a gradual variation of mechanical properties and thus minimizing the apparent property mismatch between the substrates (Yang and Shih, 1994). Such an idea of tailoring is intended for reducing the stress concentration, residual stresses and for improving the interfacial bonding strength and toughness as well, thereby alleviating the susceptibility of the bonded structures to cracking and debonding.

Recently, the solutions to related crack problems involving such graded properties were obtained by Delale and Erdogan (1988a) for the specific case of a crack in the nonhomogeneous interlayer bounded by dissimilar homogeneous half-planes, and also by Delale and Erdogan (1988b) for a crack along the interface between homogeneous and nonhomogeneous half-planes. Similar problems of an antiplane shear counterpart were considered by Erdogan and Ozturk (1992) and Ozturk and Erdogan (1993). For a crack perpendicular to the interface between two bonded nonhomogeneous media, the singular behavior of the antiplane shear stress field was examined by Erdogan (1985) and Schovanec and Walton (1988) and by Gao and Kuang (1992) under the in-plane loading. The plane strain problems of a cracked homogeneous half-plane bonded to a nonhomogeneous half-plane are due to Erdogan *et al.* (1991a) and Martin (1992). Considered by Kaw *et al.* (1992) and Bechel and Kaw (1994) are the symmetric problems of a cracked strip bonded to dissimilar materials through nonhomogeneous interfacial layers. Furthermore, Erdogan *et al.* (1991b) solved the mode III crack problem in bonded homogeneous half-planes with a nonhomogeneous interfacial zone. Consistent with the findings by Atkinson (1977), these contributions confirmed the square-root behavior of the crack-tip stress singularity that is unaffected by the existence of material nonhomogeneity.

Although the numerous investigations stated in the foregoing have resolved various issues and provided insightful results for the crack problems entailing material nonhomogeneities, most of these studies in this class appear to have been concerned with relatively simple cases of unbounded extent. The fundamental question then naturally arises as the crack-tip interaction with neighboring boundaries in such bonded media as having finite geometries, accounting for the free boundary as well as size effects. In this regard, the present paper focuses on the analysis of a plane elasticity problem of bonded dissimilar homogeneous strips with an internal crack perpendicular to the interface. A functionally graded interfacial region is assumed to exist between the two strips as a distinct transitional phase, with the corresponding nonhomogeneous elastic modulus varying continuously across its thickness in the exponential form.

As a viable and systematic method of formulating the proposed crack problem, the stiffness matrix approach is utilized circumventing the complicated and lengthy algebraic procedure involved in the analytical treatment of a layered structure. The readers are referred to papers by Kausel and Seale (1987), Choi and Thangjitham (1991, 1993, 1994), Wang and Rajapakse (1994), and Urquhart and Pindera (1994) for previous applications of this matrix approach to different types of boundary value problems. In consequence, an integral equation of the first kind with a Cauchy-type singular kernel is readily derived and solved numerically. To characterize the local mode of singular crack-tip response, the values of stress intensity factors are obtained illustrating the influence of geometric and material parameters of the bonded strips. Specifically, discussions are made with respect to the effects of the crack location and size and the strip thicknesses, in conjunction with the material nonhomogeneity in the graded interfacial zone.

2. PROBLEM STATEMENT AND BASIC EQUATIONS

The problem configuration to be investigated in this paper is illustrated in Fig. 1. As shown, an interfacial zone exists between the two dissimilar homogeneous strips to characterize the continuous transition of elastic moduli across the bimaterial interface. Such a functionally graded interfacial zone is treated as a nonhomogeneous elastic continuum of finite thickness. This bonded structure is then comprised of three infinite strips which are distinguished in order from the left-hand side, with their thicknesses and elastic moduli being h_k and E_k, $k = 1, 2, 3$, respectively. An internal crack of length $2c = b-a$ is assumed to be located in the homogeneous strip 3 perpendicular to the nominal interface with the adjacent nonhomogeneous strip where $a \geqslant 0$ and $b < h_3$. A uniform tensile strain ε_0 is applied away from and normal to the plane of the crack.

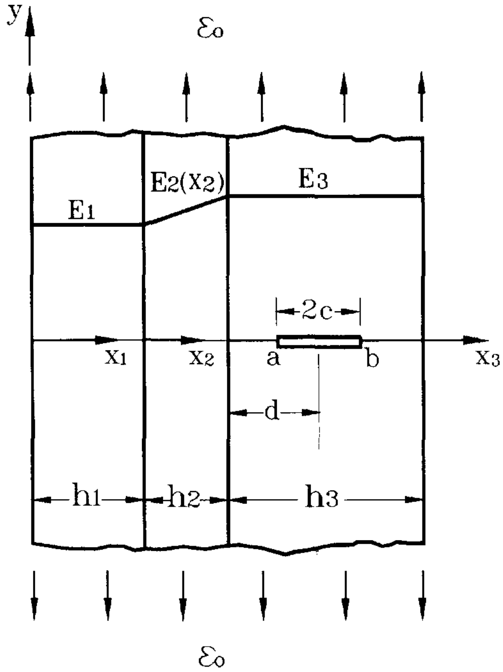

Fig. 1. Schematic representation of bonded dissimilar strips with a crack perpendicular to the functionally graded nonhomogeneous interface.

For the nonhomogeneous strip, its elastic modulus is assumed to follow an exponential variation as in the work by Erdogan *et al.* (1991a)

$$E_2(x) = E_1\, e^{\beta x} \tag{1}$$

where the nonhomogeneity parameter β is to be specified via the continuous transition of the elastic moduli through the entire width of the bonded strips. On the other hand, an assumption of the constant Poisson's ratios is made such that $v = v_k =$ constant, $k = 1, 2, 3$, based on the previous indication that neglecting the possible special variation of the Poisson's ratios within a practical range is not a very restrictive assumption (Delale and Erdogan, 1983). As will be shown, the above assumptions regarding the interfacial properties are made largely from the analytical expediency as well as physical reasoning, leading to tractable solutions.

As a result, the governing equilibrium equations with eqn (1) can be expressed as

$$\nabla^2 u_k + \frac{2}{\kappa-1}\left(\frac{\partial^2 u_k}{\partial x^2} + \frac{\partial^2 v_k}{\partial x \partial y}\right) + \frac{\beta}{\kappa-1}\left[(1+\kappa)\frac{\partial u_k}{\partial x} + (3-\kappa)\frac{\partial v_k}{\partial y}\right] = 0 \tag{2a}$$

$$\nabla^2 v_k + \frac{2}{\kappa-1}\left(\frac{\partial^2 u_k}{\partial x \partial y} + \frac{\partial^2 v_k}{\partial y^2}\right) + \beta\left(\frac{\partial u_k}{\partial y} + \frac{\partial v_k}{\partial x}\right) = 0; \quad k = 1, 2, 3 \tag{2b}$$

where $u_k(x, y)$ and $v_k(x, y)$, $k = 1, 2, 3$, are the displacement components in the x- and y-directions, respectively, $\kappa = (3-4v)$ for plane strain and $\kappa = (3-v)/(1+v)$ for plane stress, $\beta = 0$ for the homogeneous strips, i.e., $k = 1, 3$, and with reference to the local coordinates $(x, y) = (x_k, y)$, $k = 1, 2, 3$, the value of β is thus obtained as

$$\beta = \frac{1}{h_2}\ln\left(\frac{E_3}{E_1}\right). \tag{3}$$

Upon taking the geometric and loading symmetry with respect to x-axis into account, it is sufficient to consider only the upper half of the medium $y \geqslant 0$. The conditions of traction-free boundaries of the bonded strips and the continuity across the nominal interfaces are then expressed as

$$\sigma_{1xx}^{-} = 0, \quad \tau_{1xy}^{-} = 0, \quad \sigma_{3xx}^{+} = 0, \quad \tau_{3xy}^{+} = 0; \quad 0 \leqslant y < \infty \tag{4}$$

$$u_k^{+} = u_{k+1}^{-}, \quad v_k^{+} = v_{k+1}^{-}, \quad \sigma_{kxx}^{+} = \sigma_{(k+1)xx}^{-}, \quad \tau_{kxy}^{+} = \tau_{(k+1)xy}^{-}; \quad k = 1, 2, \quad 0 \leqslant y < \infty \tag{5}$$

where the superscript $-/+$ refers to the left-/right-hand surfaces of the constituent strips.

In addition, the following condition is to be met

$$\tau_{kxy}(x_k, 0) = 0; \quad 0 \leqslant x_k \leqslant h_k, \quad k = 1, 2, 3 \tag{6}$$

together with a set of mixed conditions imposed on the location of the cracked plane as

$$v_k(x_k, 0) = 0; \quad 0 \leqslant x_k \leqslant h_k, \quad k = 1, 2 \tag{7a}$$

$$v_3(x_3, 0) = 0; \quad 0 \leqslant x_3 \leqslant a, \quad b \leqslant x_3 \leqslant h_3 \tag{7b}$$

$$\sigma_{3yy}(x_3, 0) = \sigma_0(x_3); \quad a \leqslant x_3 \leqslant b \tag{7c}$$

where $\sigma_0(x_3)$ is an arbitrary function. It should be remarked that under the prescribed far-field uniform strain loading, the superposition principle renders the equivalent crack surface traction to be applied as $\sigma_0(x_3) = -E_3\varepsilon_0/(1-v^2)$ for plane strain and $\sigma_0(x_3) = -E_3\varepsilon_0$ for plane stress.

By employing the Fourier integral transform, the governing eqns (2) are first solved for the case of homogeneous strips with $\beta = 0$ and $(x, y) = (x_k, y)$, $k = 1, 3$, to obtain the general solutions of displacement components such that (Sneddon and Lowengrub, 1969).

$$u_k(x, y) = -\frac{2}{\pi}\int_0^{\infty}\left[\left(A_{k1} + xB_{k1} - \frac{\kappa}{s}B_{k2}\right)\sinh sx + \left(A_{k2} + xB_{k2} - \frac{\kappa}{s}B_{k1}\right)\cosh sx\right]\cos sy \, \mathrm{d}s$$
$$+ \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1-\kappa}{2|s|} + y\right)H_k \, \mathrm{e}^{-|s|y - isx} \, \mathrm{d}s \tag{8a}$$

$$v_k(x, y) = \frac{2}{\pi}\int_0^{\infty}[(A_{k1} + xB_{k1})\cosh sx + (A_{k2} + xB_{k2})\sinh sx]\sin sy \, \mathrm{d}s$$
$$- \frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{s}{|s|}\left(\frac{1+\kappa}{2|s|} + y\right)H_k \, \mathrm{e}^{-|s|y - isx} \, \mathrm{d}s; \quad k, 1, 3 \tag{8b}$$

and the general solutions of stress components are obtained as

$$\sigma_{kxx}(x, y) = -\frac{4\mu_k}{\pi}\int_0^{\infty}\left\{\left[s(A_{k1} + xB_{k1}) - \left(\frac{1+\kappa}{2}\right)B_{k2}\right]\cosh sx\right.$$
$$\left. + \left[s(A_{k2} + xB_{k2}) - \left(\frac{1+\kappa}{2}\right)B_{k1}\right]\sinh sx\right\}\cos sy \, \mathrm{d}s - \frac{\mu_k i}{\pi}\int_{-\infty}^{\infty}\left(sy - \frac{s}{|s|}\right)H_k \, \mathrm{e}^{-|s|y - isx} \, \mathrm{d}s \tag{9a}$$

$$\sigma_{kyy}(x, y) = \frac{4\mu_k}{\pi}\int_0^{\infty}\left\{\left[s(A_{k1} + xB_{k1}) + \left(\frac{3-\kappa}{2}\right)B_{k2}\right]\cosh sx\right.$$
$$\left. + \left[s(A_{k2} + xB_{k2}) + \left(\frac{3-\kappa}{2}\right)B_{k1}\right]\sinh sx\right\}\cos sy \, \mathrm{d}s + \frac{\mu_k i}{\pi}\int_{-\infty}^{\infty}\left(sy + \frac{s}{|s|}\right)H_k \, \mathrm{e}^{-|s|y - isx} \, \mathrm{d}s \tag{9b}$$

$$\tau_{kxy}(x,y) = \frac{4\mu_k}{\pi}\int_0^{\infty}\left\{\left[s(A_{k2}+xB_{k2})+\left(\frac{1-\kappa}{2}\right)B_{k1}\right]\cosh sx+\left[s(A_{k1}+xB_{k1})\right.\right.$$
$$\left.\left.+\left(\frac{1-\kappa}{2}\right)B_{k2}\right]\sinh sx\right\}\sin sy\,\mathrm{d}s-\frac{\mu_k}{\pi}\int_{-\infty}^{\infty}|s|yH_k\,\mathrm{e}^{-|s|y-isx}\,\mathrm{d}s; \quad k=1,3 \tag{9c}$$

where $\mu_k = E_k/2(1+\nu)$ is the shear modulus, s is the transform variable, and $A_{kj}(s)$, $B_{kj}(s)$, $H_k(s)$, $k = 1,3, j = 1,2$, are arbitrary unknowns to be evaluated. It is noted that $H_1(s) = 0$ for the uncracked strip ($k = 1$), while $H_3(s)$ for the strip with a crack ($k = 3$) is to be determined.

For the nonhomogeneous strip with $\beta \neq 0$ in eqn (3) and $(x,y) = (x_2,y)$, the general solutions of corresponding displacement and stress components are obtained as (Erdogan *et al.*, 1991a)

$$u_2(x,y) = \frac{2}{\pi}\int_0^{\infty}\sum_{j=1}^{4} m_jF_j\,\mathrm{e}^{n_jx}\cos sy\,\mathrm{d}s \tag{10a}$$

$$v_2(x,y) = \frac{2}{\pi}\int_0^{\infty}\sum_{j=1}^{4} F_j\,\mathrm{e}^{n_jx}\sin sy\,\mathrm{d}s \tag{10b}$$

$$\sigma_{2xx}(x,y) = \frac{2\mu_o\,\mathrm{e}^{\beta x}}{\pi(\kappa-1)}\int_0^{\infty}\sum_{j=1}^{4}[(1+\kappa)m_jn_j+s(3-\kappa)]F_j\,\mathrm{e}^{n_jx}\cos sy\,\mathrm{d}s \tag{10c}$$

$$\sigma_{2yy}(x,y) = \frac{2\mu_o\,\mathrm{e}^{\beta x}}{\pi(\kappa-1)}\int_0^{\infty}\sum_{j=1}^{4}[s(1+\kappa)+(3-\kappa)m_jn_j]F_j\,\mathrm{e}^{n_jx}\cos sy\,\mathrm{d}s \tag{10d}$$

$$\tau_{2xy}(x,y) = \frac{2\mu_o\,\mathrm{e}^{\beta x}}{\pi}\int_0^{\infty}\sum_{j=1}^{4}(n_j-sm_j)F_j\,\mathrm{e}^{n_jx}\sin sy\,\mathrm{d}s \tag{10e}$$

where $\mu_o = E_1/2(1+\nu)$, $F_j(s)$, $j = 1,\ldots,4$, are arbitrary unknowns, $n_j(s)$, $j = 1,\ldots,4$, are the roots of the following characteristic equation

$$(n^2+\beta n-s^2)^2+\left(\frac{3-\kappa}{1+\kappa}\right)\beta^2s^2 = 0 \tag{11}$$

from which it can be shown that

$$n_j = -\frac{1}{2}(\beta+\gamma\cos\theta)+(-1)^{j+1}\frac{i}{2}\gamma\sin\theta; \quad j = 1,2 \tag{12a}$$

$$n_j = -\frac{1}{2}(\beta-\gamma\cos\theta)+(-1)^{j+1}\frac{i}{2}\gamma\sin\theta; \quad j = 3,4 \tag{12b}$$

and $m_j(s)$, $j = 1,\ldots,4$, are given as

$$m_j = \frac{(\kappa-1)(n_j^2+\beta n_j)-(\kappa+1)s^2}{[2n_j+(\kappa-1)\beta]s}; \quad j = 1,\ldots,4 \tag{13}$$

in which $i = (-1)^{1/2}$ and $\theta(s)$ and $\gamma(s)$ are expressed as

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{4\beta s}{\beta^2+4s^2}\sqrt{\frac{3-\kappa}{1+\kappa}}\right), \quad \gamma = \left[(\beta^2+4s^2)^2+16\beta^2 s^2\left(\frac{3-\kappa}{1+\kappa}\right)\right]^{1/4}. \tag{14}$$

It can be seen that the general solutions as given in eqns (8b), (9c), (10b), and (10e) satisfy the conditions in eqns (6) and (7a) identically. In addition, upon introducing an auxiliary function which is defined in the form:

$$\phi(x_3) = \frac{\partial}{\partial x_3} v_3(x_3, 0)\,; \quad a \leqslant x_3 \leqslant b \tag{15}$$

the expression for the unknown $H_3(s)$ in eqns (8) and (9) can be written as

$$H_3(s) = -\frac{2}{1+\kappa}\int_a^b \phi(r)\,\mathrm{e}^{isr}\,\mathrm{d}r. \tag{16}$$

To obtain the expressions for a total of remaining twelve arbitrary unknowns, A_{kj}, B_{kj}, $k = 1,3, j = 1,2$, and $F_j, j = 1,\ldots,4$, involved in the general solutions of elasticity equations, a set of homogeneous boundary and interface conditions in eqns (4) and (5) can be directly applied so that a system of linear equations can be established for these unknowns. As an alternative and convenient approach to accomplishing this task, the method of stiffness matrix formulation is employed circumventing the difficulties that may arise from the above complicated and lengthy algebraic manipulation.

3. STIFFNESS MATRIX FORMULATION

As a first step in utilizing the stiffness matrix approach, the following quantities are defined

$$\bar{\mathbf{d}}_k = \{\bar{u}_k\ \bar{v}_k\}, \quad \bar{\boldsymbol{\sigma}}_k = \{\bar{\sigma}_{kxx}\ \bar{\tau}_{kxy}\}\,; \quad k = 1,2,3 \tag{17a,b}$$

where $\bar{\mathbf{d}}_k(x_k, s)$ and $\bar{\boldsymbol{\sigma}}_k(x_k, s)$ are, respectively, vectors containing the displacements and tractions of the constituting strips in the Fourier transformed domain (x_k, s).

Subsequently, in terms of A_k, B_{kj}, $k = 1,3, j = 1,2$, and $F_j, j = 1,\ldots,4$, the two vectors containing the displacements $\bar{\mathbf{d}}_k^{\mp}(s)$ and tractions $\bar{\boldsymbol{\sigma}}_k^{\mp}(s)$ evaluated at the left- $(-)$ and right-hand side $(+)$ surfaces of each strip can be expressed in matrix form. With the elimination of the unknowns between these two separate matrix equations, the local stiffness matrix equations relating the surface tractions to the corresponding displacements of strips 1 and 2 are constructed in the form as

$$\left[\begin{array}{c|c} \mathbf{K}_{11}^{(k)} & \mathbf{K}_{12}^{(k)} \\ \hline \mathbf{K}_{21}^{(k)} & \mathbf{K}_{22}^{(k)} \end{array}\right] \left\{\begin{array}{c} \bar{\mathbf{d}}_k^- \\ \hline \bar{\mathbf{d}}_k^+ \end{array}\right\} = \left\{\begin{array}{c} \bar{\boldsymbol{\sigma}}_k^- \\ \hline -\bar{\boldsymbol{\sigma}}_k^+ \end{array}\right\}; \quad k = 1,2 \tag{18}$$

and in a similar fashion, that of the cracked strip 3 can be obtained as

$$\left[\begin{array}{c|c} \mathbf{K}_{11}^{(3)} & \mathbf{K}_{12}^{(3)} \\ \hline \mathbf{K}_{21}^{(3)} & \mathbf{K}_{22}^{(3)} \end{array}\right] \left\{\begin{array}{c} \bar{\mathbf{d}}_3^- \\ \hline \bar{\mathbf{d}}_3^+ \end{array}\right\} - \left\{\begin{array}{c} \mathbf{f}_o^- \\ \hline \mathbf{f}_o^+ \end{array}\right\} = \left\{\begin{array}{c} \bar{\boldsymbol{\sigma}}_3^- \\ \hline -\bar{\boldsymbol{\sigma}}_3^+ \end{array}\right\} \tag{19}$$

where $\mathbf{K}_{lm}^k(s)$, $k = 1,2,3$, $l,m = 1,2$, are 2×2 submatrices of the 4×4 symmetric local stiffness matrices (see Appendix) which are functions of the Fourier transform variable, the elastic constants, and the geometry of the strips.

In addition, a vector $\{\mathbf{f}_o^-(s)\ \mathbf{f}_o^+(s)\}$ which is of length four as in eqn (19) is expressed as

$$\left\{\begin{array}{c}\mathbf{f}_o^- \\ \hline \mathbf{f}_o^+\end{array}\right\} = \left[\begin{array}{c|c}\mathbf{K}_{11}^{(3)} & \mathbf{K}_{12}^{(3)} \\ \hline \mathbf{K}_{21}^{(3)} & \mathbf{K}_{22}^{(3)}\end{array}\right]\left\{\begin{array}{c}\mathbf{g}_o^- \\ \hline \mathbf{g}_o^+\end{array}\right\} - 2\mu_3\left\{\begin{array}{c}\mathbf{h}_o^- \\ \hline \mathbf{h}_o^+\end{array}\right\} \tag{20}$$

where the elements of a vector $\{\mathbf{g}_o^-(s)\ \mathbf{g}_o^+(s)\} = \{g_{o1}\ g_{o2}\ g_{o3}\ g_{o4}\}$ are given as

$$g_{o1}(s) = \frac{1}{\kappa+1}\int_a^b \left(\frac{\kappa-1}{2s}+r\right) \mathrm{e}^{-sr}\phi(r)\,\mathrm{d}r \tag{21a}$$

$$g_{o2}(s) = \frac{1}{\kappa+1}\int_a^b \left(\frac{\kappa+1}{2s}-r\right) \mathrm{e}^{-sr}\phi(r)\,\mathrm{d}r \tag{21b}$$

$$g_{o3}(s) = \frac{1}{\kappa+1}\int_a^b \left[\frac{\kappa-1}{2s}+(h_3-r)\right] \mathrm{e}^{-s(h_3-r)}\phi(r)\,\mathrm{d}r \tag{21c}$$

$$g_{o4}(s) = \frac{1}{\kappa+1}\int_a^b \left[(h_3-r)-\frac{\kappa+1}{2s}\right] \mathrm{e}^{-s(h_3-r)}\phi(r)\,\mathrm{d}r \tag{21d}$$

and those of a vector $\{\mathbf{h}_o^-(s)\ \mathbf{h}_o^+(s)\} = \{h_{o1}\ h_{o2}\ h_{o3}\ h_{o4}\}$ are written as

$$h_{o1}(s) = \frac{1}{\kappa+1}\int_a^b sr\,\mathrm{e}^{-sr}\phi(r)\,\mathrm{d}r \tag{22a}$$

$$h_{o2}(s) = \frac{1}{\kappa+1}\int_a^b (1-sr)\,\mathrm{e}^{-sr}\phi(r)\,\mathrm{d}r \tag{22b}$$

$$h_{o3}(s) = \frac{1}{\kappa+1}\int_a^b s(h_3-r)\,\mathrm{e}^{-s(h_3-r)}\phi(r)\mathrm{d}r \tag{22c}$$

$$h_{o4}(s) = \frac{1}{\kappa+1}\int_a^b [s(h_3-r)-1]\,\mathrm{e}^{-s(h_3-r)}\phi(r)\,\mathrm{d}r. \tag{22d}$$

It can now be illustrated that the elements of stiffness matrices for the homogeneous strips 1 and 3 are all real, while those for the nonhomogeneous strip are also real such that

$$\mathrm{Im}\,\mathbf{K}_{lm}^{(2)}(s) = 0 \tag{23}$$

together with the following asymptotic behavior as the Fourier variable s approaches infinity:

$$\lim_{s\to\infty}\frac{1}{s}\mathbf{K}_{lm}^{(k)}(s) = \begin{cases}\mathbf{K}_{ll\infty}^{(k)}; & l = m, \quad k = 1,2,3 \\ 0; & l \neq m, \quad k = 1,2,3\end{cases} \tag{24}$$

where $\mathbf{K}_{ll\infty}^{(k)}$, $k = 1,2,3$, denote 2×2 symmetric submatrices containing limiting nonzero values.

After defining $\bar{\boldsymbol{\delta}}_{k+1}(s) \equiv \bar{\mathbf{d}}_k^+(s) = \bar{\mathbf{d}}_{k+1}^-(s)$, $k = 1,2$, as vectors for the common interfacial displacements between the adjacent strips and $\bar{\boldsymbol{\delta}}_1(s) \equiv \bar{\mathbf{d}}_1^-(s)$ and $\bar{\boldsymbol{\delta}}_4(s) \equiv \bar{\mathbf{d}}_3^+(s)$, successive applications of boundary and interface conditions in eqns (4) and (5) to the local matrix equations in eqns (18) and (19) result in a system of global stiffness matrix equations for the three-layer bonded structure to be assembled as

$$\mathbf{K}_{11}^{(1)}\bar{\boldsymbol{\delta}}_1+\mathbf{K}_{12}^{(1)}\bar{\boldsymbol{\delta}}_2=0 \tag{25a}$$

$$\mathbf{K}_{21}^{(1)}\bar{\boldsymbol{\delta}}_1+(\mathbf{K}_{22}^{(1)}+\mathbf{K}_{11}^{(2)})\bar{\boldsymbol{\delta}}_2+\mathbf{K}_{12}^{(2)}\bar{\boldsymbol{\delta}}_3=0 \tag{25b}$$

$$\mathbf{K}_{21}^{(2)}\bar{\boldsymbol{\delta}}_2+(\mathbf{K}_{22}^{(2)}+\mathbf{K}_{11}^{(3)})\bar{\boldsymbol{\delta}}_3+\mathbf{K}_{12}^{(3)}\bar{\boldsymbol{\delta}}_4=\mathbf{f}_o^- \tag{25c}$$

$$\mathbf{K}_{21}^{(3)}\bar{\boldsymbol{\delta}}_3+\mathbf{K}_{22}^{(3)}\bar{\boldsymbol{\delta}}_4=\mathbf{f}_o^+. \tag{25d}$$

The above system of algebraic equations is expressed in contracted notation as

$$\mathbf{K}\bar{\boldsymbol{\delta}}=\mathbf{f} \tag{26}$$

where $\mathbf{K}(s)$ is a banded and symmetric global stiffness matrix of order 8×8 which is numerically stable, $\bar{\boldsymbol{\delta}}(s)$ is a global vector for the interfacial displacements as the basic unknown variables $\{\bar{\boldsymbol{\delta}}_1\ \bar{\boldsymbol{\delta}}_2\ \bar{\boldsymbol{\delta}}_3\ \bar{\boldsymbol{\delta}}_4\}$, and $\mathbf{f}(s)$ is a vector containing the auxiliary function ϕ and zero elements such that $\{0\ 0\ \mathbf{f}_o^-\ \mathbf{f}_o^-\}$. These vectors are of eight units in length.

With the global interfacial displacement vector $\bar{\boldsymbol{\delta}}$ obtained by solving the global matrix equation (26), the required unknowns, A_{kj}, B_{kj}, $k=1,3$, $j=1,2$, and F_j, $j=1,\ldots,4$, in the general solutions of the current elasticity problem that satisfy the prescribed conditions in eqns (4) and (5) can be expressed in terms of the local interfacial displacements $\bar{\boldsymbol{\delta}}_k$, $k=1,\ldots,4$, in the straightforward manner. The auxiliary function then remains as the only unknown that should be determined from the mixed conditions on the crack surface in eqn (7c).

4. SINGULAR INTEGRAL EQUATION

In conjunction with the expression given in eqn (16), the traction component σ_{3yy} acting on the cracked plane $y=0$ can be written as

$$\begin{aligned}\sigma_{3yy}(x,0) = &-\frac{2\mu_3 i}{\pi(1+\kappa)}\int_a^b \phi(r)\,\mathrm{d}r\int_{-\infty}^{\infty}\operatorname{sgn}(s)\mathrm{e}^{is(r-x)}\,\mathrm{d}s\\ &+\frac{4\mu_3}{\pi}\int_0^{\infty}\left\{\left[s(A_{31}+xB_{31})+\left(\frac{3-\kappa}{2}\right)B_{32}\right]\cosh sx\right.\\ &\left.+\left[s(A_{32}+xB_{32})+\left(\frac{3-\kappa}{2}\right)B_{31}\right]\sinh sx\right\}\mathrm{d}s;\quad 0\leqslant x=x_3\leqslant h_3\end{aligned} \tag{27}$$

where sgn( ) denotes the signum function.

As outlined in the foregoing, the expressions for A_{3j} and B_{3j}, $j=1,2$, are obtained from the relationship between these unknowns and the local interfacial displacements $\bar{\boldsymbol{\delta}}_3=\bar{\mathbf{d}}_3^-$ and $\bar{\boldsymbol{\delta}}_4=\bar{\mathbf{d}}_3^+$ such that

$$A_{3j}(s)=\sum_{k=1}^{4}(L_{jk}g_{ok}-2\mu_3 M_{jk}h_{ok});\quad j=1,2 \tag{28a}$$

$$B_{3j}(s)=\sum_{k=1}^{4}(L_{(j-2)k}g_{ok}-2\mu_3 M_{(j+2)k}h_{ok});\quad j=1,2 \tag{28b}$$

where $L_{jk}(s)$ and $M_{jk}(s)$, $j,k=1,\ldots,4$, are written as

$$L_{jk}(s)=-\alpha_{jk}+\sum_{l=1}^{4}\alpha_{jl}\Gamma_{lk},\quad M_{jk}(s)=\sum_{l=1}^{4}\alpha_{jl}F_{(l+4)(k+4)};\quad j,k=1,2,3,4 \tag{29}$$

in which $\Gamma_{lk}(s)$, $l,k=1,\ldots,4$, are expressed as

$$\Gamma_{lk}(s) = \sum_{j=1}^{4} F_{(l+4)(j+4)} K_{jk}^{(3)}\,; \quad l,k = 1,2,3,4 \tag{30}$$

the function $F_{lk}(s)$, $l, k = 1, \ldots, 8$, denotes the elements of the global flexibility matrix $\mathbf{F}(s) \equiv \mathbf{K}^{-1}(s)$, and the expressions for $\alpha_{jk}(s), j, k = 1, \ldots, 4$, are given in the Appendix.

The unknown auxiliary function ϕ in eqn (27) is to be evaluated by applying the crack surface condition in eqn (7c), which has not yet been satisfied. As a result, upon substituting eqn (28) into eqn (27) and recalling the Fourier representation of a generalized function (Friedman, 1969)

$$\int_{-\infty}^{\infty} \operatorname{sgn}(s)\, \mathrm{e}^{isr}\, \mathrm{d}s = \frac{2i}{r} \tag{31}$$

a singular integral equation of the first kind is derived as

$$\int_a^b \left[\frac{1}{r-x} + p(x,r)\right]\phi(r)\, \mathrm{d}r = \frac{\pi(1+\kappa)}{4\mu_3}\sigma_o(x)\,; \quad a \leqslant x \leqslant b \tag{32}$$

where the kernel $p(x, r)$ is expressed as

$$p(x,r) = \int_0^{\infty} \Lambda(s,x,r)\, \mathrm{d}s \tag{33}$$

in which the integrand $\Lambda(s, x, r)$ is written as

$$\begin{aligned}\Lambda(s,x,r) = &\left[\left(\frac{\kappa-1}{2s}+r\right)f_1(x,s) + \left(\frac{1+\kappa}{2s}-r\right)f_2(x,s)\right]\mathrm{e}^{-sr} \\ &+\left[\left(\frac{\kappa-1}{2s}+h_3-r\right)f_3(x,s) + \left(h_3-r-\frac{1+\kappa}{2s}\right)f_4(x,s)\right]\mathrm{e}^{-s(h_3-r)} \\ &-2\mu_3[srg_1(x,s)+(1-sr)g_2(x,s)]\,\mathrm{e}^{-sr} \\ &-2\mu_3[s(h_3-r)g_3(x,s)-(1-sh_3+sr)g_4(x,s)]\,\mathrm{e}^{-s(h_3-r)}\end{aligned} \tag{34}$$

together with the expressions for $f_j(x, s)$ and $g_j(x, s)$, $j = 1, \ldots, 4$ defined as

$$\begin{aligned}\begin{Bmatrix} f_j(x,s) \\ g_j(x,s)\end{Bmatrix} = &\begin{Bmatrix} L_{1j} \\ M_{1j}\end{Bmatrix} s\cosh sx + \begin{Bmatrix} L_{2j} \\ M_{2j}\end{Bmatrix} s\sinh sx + \begin{Bmatrix} L_{3j} \\ M_{3j}\end{Bmatrix}\left(sx\cosh sx + \frac{3-\kappa}{2}\sinh sx\right) \\ &+\begin{Bmatrix} L_{4j} \\ M_{4j}\end{Bmatrix}\left(sx\sinh sx + \frac{3-\kappa}{2}\cosh sx\right); \quad j = 1,2,3,4.\end{aligned} \tag{35}$$

It should be mentioned that when the crack is within the homogeneous strip such that $a > 0$ and $b < h_3$, the integrand in eqn (34) possesses the following asymptotic behavior for large values of s

$$\lim_{s\to\infty} \Lambda(s,x,r) = 0\,; \quad a \leqslant (x,r) \leqslant b \tag{36}$$

so that the function $p(x, r)$ can be referred to as a Fredholm kernel bounded for all values of x and r in their closed domains of definition $[a\,b]$ and only a simple Cauchy-type kernel $1/(r-x)$ in eqn (32) contributes to the singular behavior of the solution to the integral equation (Muskhelishvili, 1953).

For the crack tip intersecting the nominal interface with the nonhomogeneous strip as $a = 0$ and $b < h_3$, there exist logarithmic singularities in the kernel $p(x, r)$ in eqn (33) as x and r approach zero simultaneously. As shown by Erdogan *et al.* (1991a,b), the logarithmic singularities are, however, square integrable that do not have any influence on the crack-tip singularity. Such additional unbounded terms can thus be treated as a Fredholm kernel, and the singular kernel in this limiting case is yet attributed to the same Cauchy type.

Because of the presence of the dominant Cauchy-type kernel in eqn (32) as the sole contribution to the singular nature of the auxiliary function ϕ, the crack-tip behavior is characterized by the standard square-root singularity for both the crack-tip locations $a > 0$ and $a = 0$. In the normalized intervals given as

$$x = \frac{b-a}{2}\xi + \frac{b+a}{2}, \quad r = \frac{b-a}{2}\eta + \frac{b+a}{2}; \quad -1 < (\xi, \eta) < 1 \tag{37}$$

preserving the correct nature of the problem singularity, the solution to the integral equation can therefore be of the form (Muskhelishvili, 1953)

$$\phi(r) = \Phi(\eta) = \frac{H(\eta)}{\sqrt{1-\eta^2}}; \quad |\eta| < 1 \tag{38}$$

where $H(\eta)$ is an unknown bounded function and nonzero at $\eta = \pm 1$ as

$$H(\eta) = \sum_{n=0}^{\infty} c_n T_n(\eta); \quad |\eta| < 1 \tag{39}$$

in which c_n, $n \geqslant 0$, are the constants to be evaluated, T_n is the Chebyshev polynomial of the first kind, and $1/(1-\eta^2)^{1/2}$ is the corresponding weight function. It can be shown that provided $c_o = 0$, the above series expansion and the orthogonality of T_n satisfy the single-valuedness condition expressed as

$$\int_{-1}^{1} \Phi(\eta)\, \mathrm{d}\eta = 0. \tag{40}$$

Upon substituting eqns (37)–(39) into eqn (32), truncating the series at $n = N$, and using the integral formulas (Gradshteyn and Ryzhik, 1980)

$$\frac{1}{\pi}\int_{-1}^{1} \frac{T_n(\eta)\, \mathrm{d}\eta}{\sqrt{1-\eta^2}(\eta-\xi)} = U_{n-1}(\xi); \quad n \geqslant 1, \quad |\xi| < 1 \tag{41}$$

where U_n is the Chebyshev polynomial of the second kind, the singularity of the integral equation is removed such that

$$\sum_{n=1}^{N} c_n \left[\pi U_{n-1}(\xi) + \int_{-1}^{1} \frac{p_*(\xi, \eta) T_n(\eta)}{\sqrt{1-\eta^2}} \mathrm{d}\eta\right] = \frac{\pi(1+\kappa)}{4\mu_3}\sigma_o(\xi); \quad |\xi| < 1 \tag{42}$$

where $p_*(\xi, \eta) = (b-a)p(x, r)/2$.

To recast the above functional equations into a solvable form for c_n, $1 \leqslant n \leqslant N$, the collocation technique is applied. The roots of T_N are then selected as a set of collocation points which are concentrated near the end singular points $\xi = \pm 1$

$$T_N(\xi_i) = 0, \quad \xi_i = \cos\theta_i, \quad \theta_i = \frac{\pi}{2}\frac{(2i-1)}{N}; \quad i = 1, 2, \ldots, N \tag{43}$$

and a system of N linear algebraic equations for c_n, $1 \leqslant n \leqslant N$, is obtained by substituting the collocation points $\xi = \xi_i$, $1 \leqslant i \leqslant N$, into eqn (42).

As noted, the number of terms N in eqn (39) or (42) is equal to that of collocation points ξ_i in eqn (43). The number N must be large enough so that the resulting solution is within a required degree of accuracy, with the integrals in eqns (33) and (42) evaluated by the Gauss–Legendre and Gauss–Chebyshev quadrature rules, respectively.

After the constants c_n being evaluated, the left-hand side of the integral equation in eqn (32) provides the expression for $\sigma_{3yy}(\xi, 0)$ outside $|\xi| > 1$ as well as inside $|\xi| < 1$ the crack. Thus, from eqns (32) and (37)–(39), the singular traction ahead of the crack-tip $|\xi| > 1$ can be obtained as

$$\begin{aligned} \frac{1+\kappa}{4\mu_3}\sigma_{3yy}(\xi, 0) &= \frac{1}{\pi}\sum_{n=1}^{N} c_n \int_{-1}^{1} \frac{T_n(\eta)}{\sqrt{1-\eta^2}}\left[\frac{1}{\eta-\xi} + p_*(\xi,\eta)\right]\mathrm{d}\eta \\ &= -\sum_{n=1}^{N} c_n \frac{[\xi - \mathrm{sgn}(\xi)\sqrt{\xi^2-1}]^n}{\mathrm{sgn}(\xi)\sqrt{\xi^2-1}} + \frac{1}{\pi}\sum_{n=1}^{N} c_n \int_{-1}^{1} \frac{p_*(\xi,\eta)T_n(\eta)}{\sqrt{1-\eta^2}}\mathrm{d}\eta; \quad |\xi| > 1 \end{aligned} \tag{44}$$

where the second term in the right-hand side is a higher order nonsingular term.

As the local crack-tip parameter in linear elastic fracture mechanics, the stress intensity factors are defined from the foregoing structure of the singular traction and can be evaluated in terms of the solution to the integral equation:

$$K_a \equiv \lim_{x\to a}\sqrt{2(a-x)}\,\sigma_{3yy}(x, 0) = \frac{4\mu_3}{1+\kappa}\sqrt{\frac{b-a}{2}}\sum_{n=1}^{N}(-1)^n c_n; \quad x < a \tag{45a}$$

$$K_b \equiv \lim_{x\to b}\sqrt{2(x-b)}\,\sigma_{3yy}(x, 0) = -\frac{4\mu_3}{1+\kappa}\sqrt{\frac{b-a}{2}}\sum_{n=1}^{N} c_n; \quad x > b \tag{45b}$$

where K_a and K_b denote the mode I stress intensity factors at the left- and right-hand side crack tips, respectively.

To be mentioned now is that due to the continuity of elastic moduli through the transitional interfacial zone, the cleavage stresses are also continuous at the nominal interface between the homogeneous and nonhomogeneous strips such that $\sigma_{2yy}^{+}(y) = \sigma_{3yy}^{-}(y)$, $y \geqslant 0$. As a result, the above definition of the stress intensity factors can be validated for the limiting case of $a = 0$ as well.

5. RESULTS AND DISCUSSION

To obtain the numerical results, the bonded structure and the crack geometry are specified as $h_1/h_3 = 1.0$, $h_2/h_3 = 0.25$, and $2c/h_3 = 0.5$, unless otherwise stated, and the Poisson's ratio is assumed as $\nu = \nu_k = 0.3$, $k = 1, 2, 3$. In addition, the crack surface traction in eqn (7c) is applied as $\sigma_o(x_3) = -p_o = -E_3\varepsilon_o/(1-\nu^2)$ under the plane strain state. No more than twenty terms in eqn (42) are then found to be necessary in obtaining the highly accurate values of stress intensity factors, with $K_o = p_o c^{1/2}$ as a normalizing factor.

In Figs 2a and 2b, the variations of normalized stress intensity factors K_a and K_b at the left- and right-hand side crack-tips a and b, respectively, are shown as a function of crack location d/c for different elastic moduli ratios E_3/E_1. Upon comparing these two

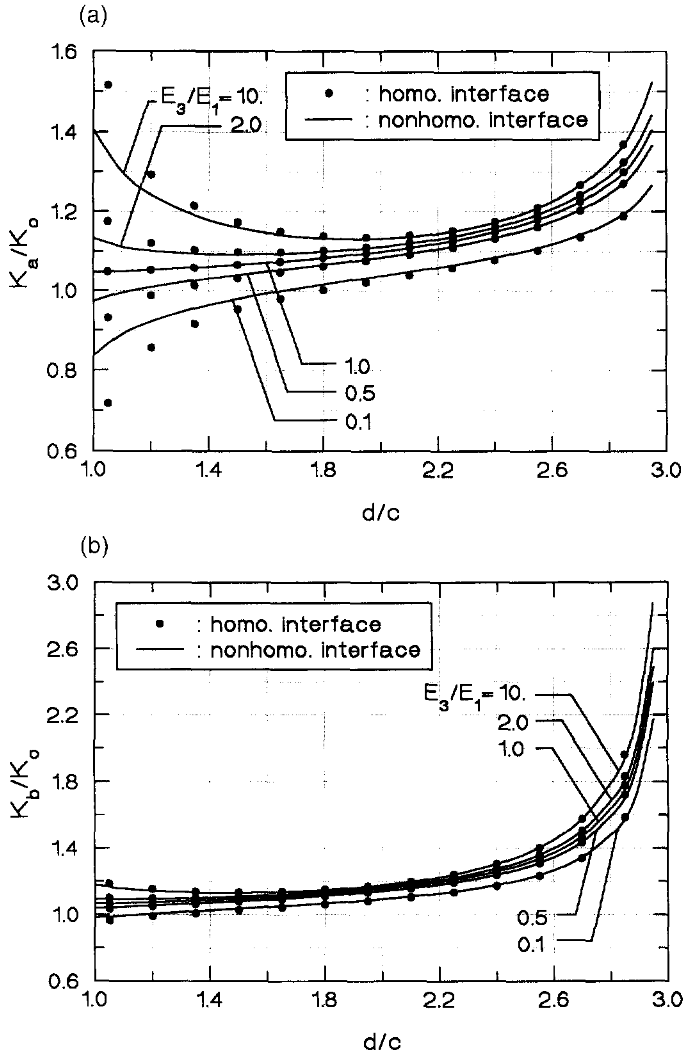

Fig. 2. Variation of normalized stress intensity factors (a) K_a/K_o and (b) K_b/K_o as a function of crack location d/c for different elastic moduli ratios E_3/E_1 ($K_o = p_o c^{1/2}$, $h_1/h_3 = 1.0$, $h_2/h_3 = 0.25$, $2c/h_3 = 0.5$).

figures, it is shown that the values of K_a are more markedly affected by the ratios E_3/E_1 than those of K_b. More specifically, Fig. 2a illustrates that the values of K_a for $E_3/E_1 \leqslant 1$ increase monotonically with d/c because of the diminishing influence exerted by the adjacent stiffer materials on the crack tip. On the other hand, those for $E_3/E_1 > 1$ attain their minima at some points of d/c before increasing further. At the other crack tip, the values of K_b in Fig. 2b increase moderately with d/c for all given ratios of E_3/E_1 and then, as expected, experience abrupt increases as the crack tip approaches the nearby free boundary. This is indicative of the fact that the behavior of the right-hand side crack tip is mainly dominated by the size of the uncracked ligament $(h_3 - b)$, while that of the left-hand side crack tip is, to a larger extent, controlled by the elastic moduli of adjacent uncracked strips.

Additionally plotted and compared in Figs 2a and 2b by circles are the normalized stress intensity factors evaluated for the homogenized interface model based on the following functionally averaged nonhomogeneous elastic modulus of the interfacial zone:

$$E_{\mathrm{avg}} = \frac{1}{h_2}\int_0^{h_2} E_o\, \mathrm{e}^{\beta x}\, \mathrm{d}x = \frac{E_o}{\beta h_2}(\mathrm{e}^{\beta h_2} - 1). \tag{46}$$

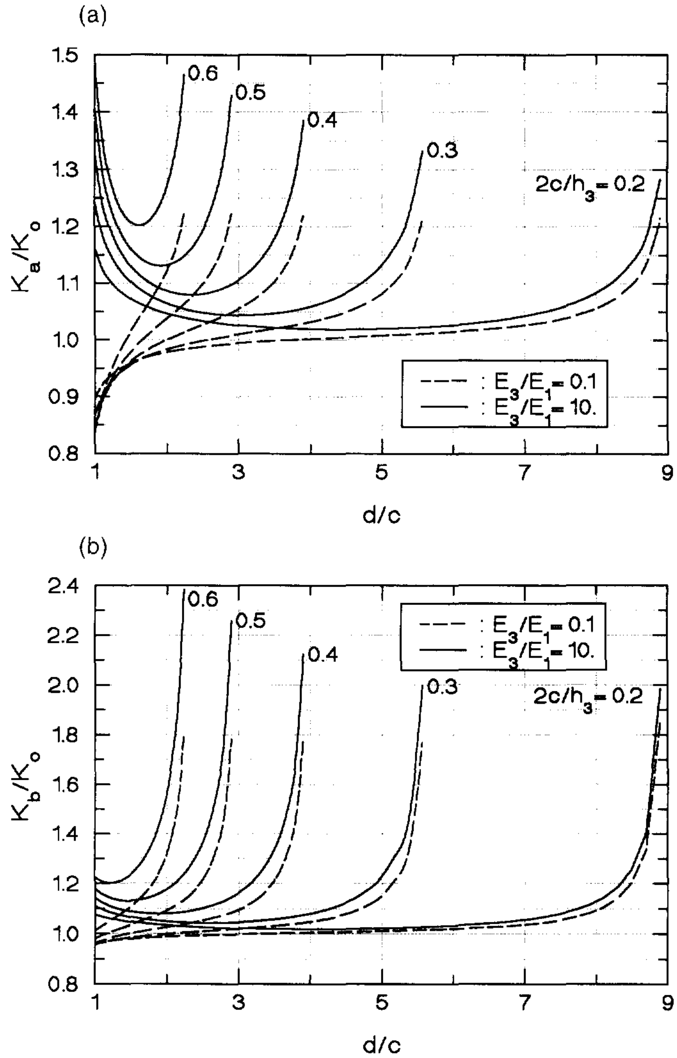

Fig. 3. Effect of crack location d/c on normalized stress intensity factors (a) K_a/K_o and (b) K_b/K_o for different crack size $2c/h_3$ and elastic moduli ratios E_3/E_1 ($K_o = p_o c^{1/2}$, $h_1/h_3 = 1.0$, $h_2/h_3 = 0.25$).

To be observed herein are the more pronounced deviations of the stress intensity factor K_a from the current nonhomogeneous interface model when the difference between elastic moduli E_1 and E_3 becomes greater and the crack advances toward the plane of the nominal interface. Otherwise, an obvious but important fact which may be stated from Fig. 2a is that the solutions to both interface models become close to each other and that the similar results are obtained for the case of K_b in Fig. 2b. The discrepancy between these two solutions, however, appears to be more notable than the case of a crack located in a semi-infinite substrate as most recently investigated by Choi (1996). It is now worthwhile to mention that when the crack tip terminates at or intersects the ideal interface with the homogenized interfacial zone with $d/c = 1$, the discrete nature of the elastic moduli renders the nonsquare-root crack-tip singularity to be obtained (Cook and Erdogan, 1972). Hence, such incompatible results corresponding to this ideal interface are not given.

The effects of crack locations are further presented in Figs 3a and 3b in conjunction with different crack sizes $2c/h_3$ for $E_3/E_1 = 0.1$ and 10. Of particular interest in these figures are also the smooth gradients of K_a and K_b for a practical range of $3.0 < d/c < 7.0$, especially when $2c/h_3 = 0.2$, prior to sharp increases and the lesser influence of E_3/E_1 for the relatively smaller crack size.

With the crack located directly beneath the nonhomogeneous interfacial zone such that $d/c = 1$, Figs 4a and 4b next show the effects of the crack size $2c/h_3$ for various elastic

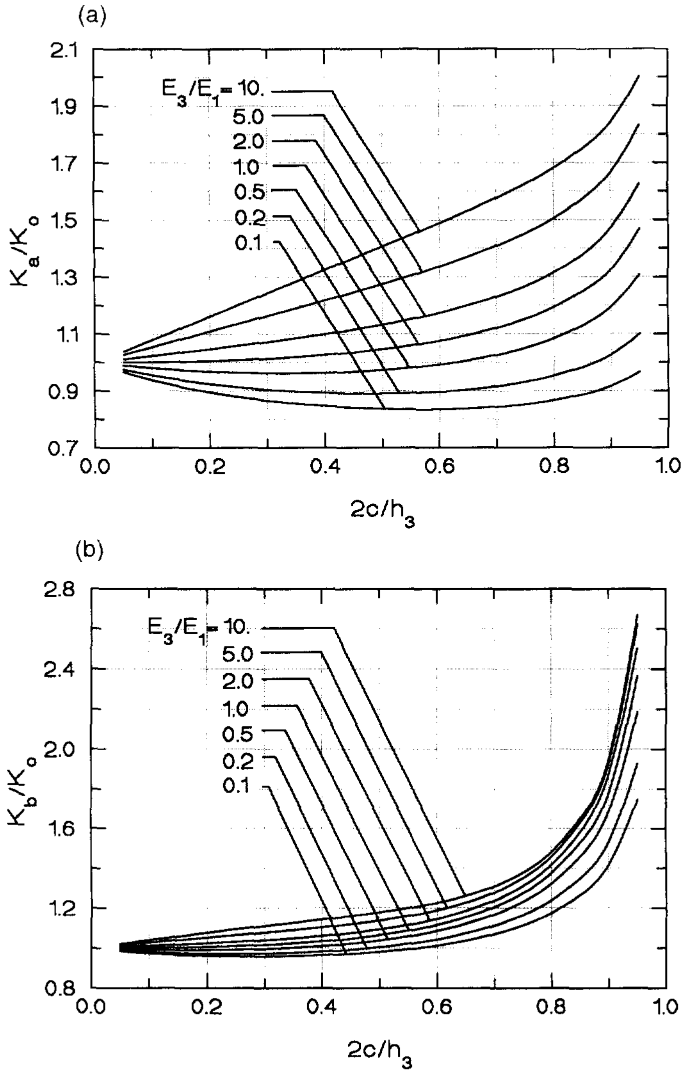

Fig. 4. Variation of normalized stress intensity factors (a) K_a/K_o and (b) K_b/K_o as a function of crack size $2c/h_3$ for different elastic moduli ratios E_3/E_1 ($K_o = p_o c^{1/2}$, $h_1/h_3 = 1.0$, $h_2/h_3 = 0.25$, $d/c = 1.0$).

moduli ratios E_3/E_1. A generic trend of monotonic increases of the normalized stress intensity factors is thus illustrated as a function of $2c/h_3$, except for slightly decreasing behavior of K_a for $E_3/E_1 < 1.0$ which is possibly due to the stiffening by the adjacent stiffer strips. Another feature observed is that as $2c/h_3$ approaches zero, the normalized stress intensity factors at both the crack tips tend to unity, implying that the tip behavior in an unbounded bonded medium is negligibly affected by the presence of the nonhomogeneous interfacial zone.

Figures 5a and 5b show that, for the crack location $d/c = 1$ and size $2c/h_3 = 0.5$, the increase in the thickness of the uncracked strip by h_1/h_3 and the decrease in the moduli ratio by E_3/E_1 lead to a substantial reduction in the values of stress intensity factors at both the crack tips. In other words, the crack-tip shielding and crack growth stabilization can be achieved by increasing the rigidity or thickness of the uncracked tip.

The variations of normalized stress intensity factors K_a and K_b with the thickness of the nonhomogeneous interfacial zone h_2/h_3 are plotted in Figs 6a and 6b, respectively. It should be pointed out that given the same relative crack size and location as in Figs 5a and 5b, the effect of increasing the thickness of the nonhomogeneous strip by h_2/h_3 is predicted in a quite different manner depending on the values of the ratio E_3/E_1. It is illustrated in

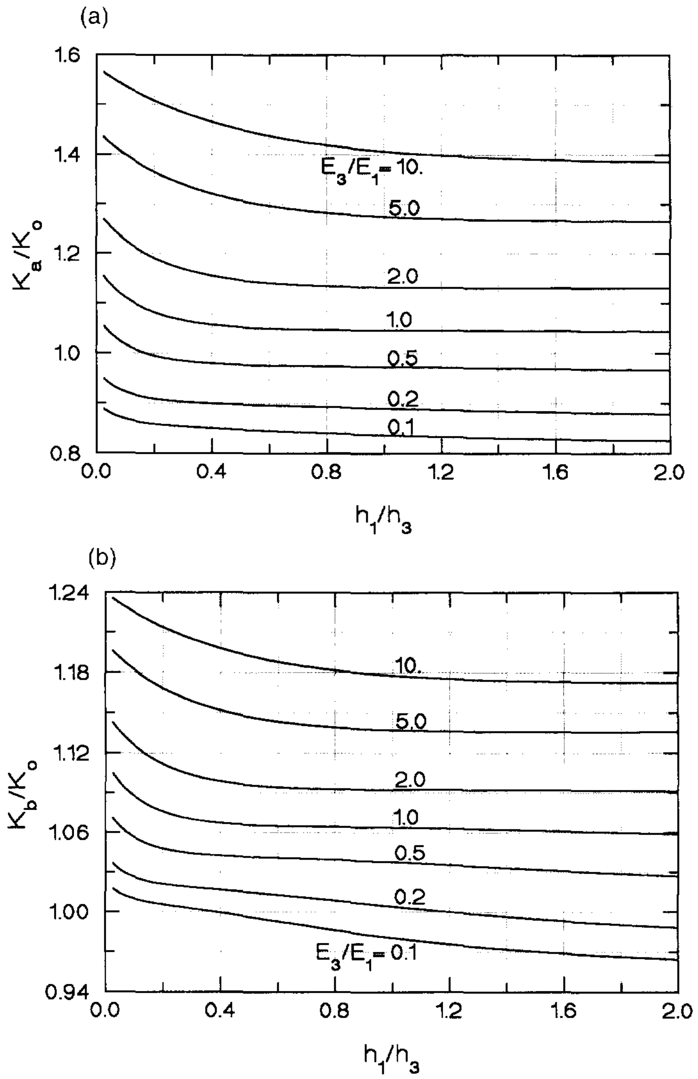

Fig. 5. Variation of normalized stress intensity factors (a) K_a/K_o and (b) K_b/K_o as a function of thickness of a homogeneous strip h_1/h_3 for different elastic moduli ratios E_3/E_1 ($K_o = p_o c^{1/2}$, $h_2/h_3 = 0.25$, $d/c = 1.0$, $2c/h_3 = 0.5$).

these figures that the spreadout of the rigidity of the nonhomogeneous interfacial zone by increasing h_2/h_3 results in a gradual increase in the magnitude of stress intensity factors for $E_3/E_1 < 1$, while the reverse behavior prevails for $E_3/E_1 > 1$. Together with the results in Figs 5a and 5b, the aforementioned crack-tip response may provide an idea as to determining the geometric and material conditions that would enhance the fail-safe capability of diffusion bonded structures from the viewpoint of fracture mechanics.

6. CONCLUDING REMARKS

An analysis has been performed to investigate the crack-tip behavior in bonded strips in the presence of a functionally graded, nonhomogeneous interfacial zone. Via the continuity of elastic moduli at the nominal interfaces, the standard order of square-root singularity was retained for the case of the crack-tip that intersects the interface, and no difficulties were thus imposed in properly applying the linear elastic fracture mechanics concepts. As a consequence, the stress intensity factors were readily evaluated which were shown to be strongly affected by various geometric parameters of the bonded media, in

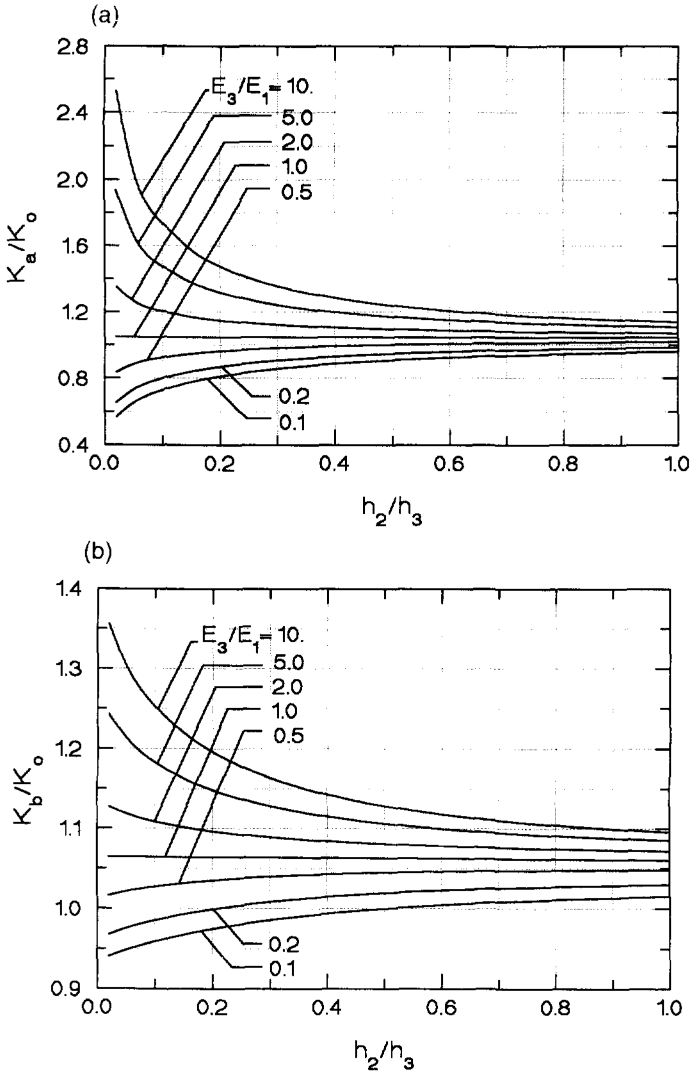

Fig. 6. Variation of normalized stress intensity factors (a) K_a/K_o and (b) K_b/K_o as a function of thickness of a nonhomogeneous strip h_2/h_3 for different elastic moduli ratios E_3/E_1 ($K_o = p_o c^{1/2}$, $h_1/h_3 = 1.0$, $d/c = 1.0$, $2c/h_3 = 0.5$).

conjunction with the material nonhomogeneity in the interfacial zone. The results based on the use of the homogenized interfacial elastic modulus were also presented as a function of varying crack locations to measure the degree of correspondence with those of the current nonhomogeneous interface model. On the basis of this study, it may be stated that the analysis of the cracked bonded structure involving the nonhomogeneous interfacial zone provides accurate information in examining the singular response at the crack-tip, particularly when the crack is very close to or terminates at the nominal interface.

REFERENCES

Atkinson, C. (1977). On stress singularities and interfaces in linear elastic fracture mechanics. *Int. J. Fract.* **13**, 807–820.

Bechel, V. T. and Kaw, A. K. (1994). Fracture mechanics of composites with nonhomogeneous interphases and nondilute fiber volume fractions. *Int. J. Solids Structures* **31**, 2053–2070.

Choi, H. J. (1996). An analysis of cracking in a layered medium with a functionally graded nonhomogeneous interface. *J. Appl. Mech.* (In press).

Choi, H. J. and Thangjitham, S. (1991). Stress analysis of multilayered anisotropic elastic media. *J. Appl. Mech.* **58**, 382–387.

Choi, H. J. and Thangjitham, S. (1993). Thermally induced interlaminar crack-tip singularities in laminated anisotropic composites. *Int. J. Fract.* **60**, 327–347.

Choi, H. J. and Thangjitham, S. (1994). The interlaminar crack-tip response in a fiber-reinforced composite laminate. *Int. J. Fract.* **66,** 121–138.
Cook, T. S. and Erdogan, F. (1972). Stresses in bonded materials with a crack perpendicular to the interface. *Int. J. Engng Sci.* **10,** 677–697.
Delale, F. and Erdogan, F. (1983). The crack problem for a nonhomogeneous plane. *J. Appl. Mech.* **50,** 609–614.
Delale, F. and Erdogan, F. (1988a). On the mechanical modeling of the interfacial region in bonded half-planes. *J. Appl. Mech.* **55,** 317–324.
Delale, F. and Erdogan, F. (1988b). Interface crack in a nonhomogeneous elastic medium. *Int. J. Engng Sci.* **26,** 559–568.
England, A. H. (1965). A crack between dissimilar media. *J. Appl. Mech.* **32,** 400–402.
Erdogan, F. (1985). The crack problem for bonded nonhomogeneous materials under antiplane shear loading. *J. Appl. Mech.* **52,** 823–828.
Erdogan, F., Kaya, A. C. and Joseph, P. F. (1991a). The crack problem in bonded nonhomogeneous materials. *J. Appl. Mech.* **58,** 410–418.
Erdogan, F., Kaya, A. C. and Joseph, P. F. (1991b). The mode III crack problem in bonded materials with a nonhomogeneous interfacial zone. *J. Appl. Mech.* **58,** 419–427.
Erdogan, F. and Ozturk, M. (1992). Diffusion problems in bonded nonhomogeneous materials with an interface cut. *Int. J. Engng Sci.* **30,** 1507–1523.
Friedman, B. (1969). *Lectures on Application-Oriented Mathematics*, Holden-Day, San Francisco.
Gao, X. and Kuang, Z.-B. (1992). Mode I fracture in two dissimilar functional nonhomogeneous planes. *Engng Fract. Mech.* **42,** 33–44.
Gradshteyn, I. S. and Ryzhik, I. M. (1980). *Table of Integrals, Series, and Products*. Academic Press, New York.
Gupta, G. D. (1973). A layered composite with a broken laminate. *Int. J. Solids Structures* **9,** 1141–1154.
Kausel, E. and Seale, S. H. (1987). Static loads in layered halfspaces. *J. Appl. Mech.* **54,** 403–408.
Kaw, A. K., Selvarathinam, A. S. and Besterfield, G. H. (1992). Comparison of interphase models for a crack in fiber reinforced composite. *Theoret. Appl. Fract. Mech.* **17,** 133–147.
Martin, P. A. (1992). Tip behaviour for cracks in bonded inhomogeneous materials. *J. Engng Math.* **26,** 467–480.
Muskhelishvili, N. I. (1953). *Singular Integral Equations*. Noordhoff, Groningen, The Netherlands.
Ozturk, M. and Erdogan, F. (1993). Antiplane shear crack problem in bonded materials with a graded interfacial zone. *Int. J. Engng Sci.* **31,** 1641–1657.
Rice, J. R. and Sih, G. C. (1965). Plane problems of cracks in dissimilar material. *J. Appl. Mech.* **32,** 418–423.
Schovanec, L. and Walton, J. R. (1988). On the order of stress singularity for an antiplane shear crack at the interface of two bonded inhomogeneous elastic materials. *J. Appl. Mech.* **55,** 234–236.
Sneddon, I. N. and Lowengrub, M. (1969). *Crack Problems in the Classical Theory of Elasticity*. John Wiley & Sons, Inc., New York.
Urquhart, E. E. and Pindera, M.-J. (1994). Incipient separation between a frictionless flat punch and an anisotropic multilayered half plane. *Int. J. Solids Structures* **31,** 2445–2461.
Wagner, T., Kirchheim, R. and Rühle, M. (1995). Chemical reactions at metal/ceramic interfaces during diffusion bonding. *Acta Metall. Mat.* **43,** 1053–1063.
Wang, Y. and Rajapakse, R. K. N. D. (1994). An exact stiffness method for elastodynamics of a layered orthotropic half-plane. *J. Appl. Mech.* **61,** 339–348.
Yang, W. and Shih, C. F. (1994). Fracture along an interlayer. *Int. J. Solids Structures* **31,** 985–1002.
Zak, A. R. and Williams, M. L. (1963). Crack point stress singularities at a bimaterial interface. *J. Appl. Mech.* **30,** 142–143.

APPENDIX

The structure of the strip local stiffness matrices in eqns (18) and (19) is written as

$$\left[\begin{array}{c|c} \mathbf{K}_{11}^{(k)} & \mathbf{K}_{12}^{(k)} \\ \hline \mathbf{K}_{21}^{(k)} & \mathbf{K}_{22}^{(k)} \end{array}\right] = \left[\begin{array}{cc|cc} K_{11}^{(k)} & K_{12}^{(k)} & K_{13}^{(k)} & K_{14}^{(k)} \\ K_{12}^{(k)} & K_{22}^{(k)} & K_{23}^{(k)} & K_{24}^{(k)} \\ \hline K_{13}^{(k)} & K_{23}^{(k)} & K_{33}^{(k)} & K_{34}^{(k)} \\ K_{14}^{(k)} & K_{24}^{(k)} & K_{34}^{(k)} & K_{44}^{(k)} \end{array}\right] \quad ; \quad k = 1, 2, 3$$

and the functions $\alpha_{jk}(s), j, k = 1, \dots, 4$, in eqn (29) are expressed as

$$\alpha_{11} = \alpha_{13} = \alpha_{14} = 0, \qquad \alpha_{12} = 1, \qquad \alpha_{21} = -\frac{h_3^2}{\Delta},$$

$$\alpha_{22} = -\frac{\kappa}{s\Delta}\left(h_3 - \frac{\kappa}{2s}\sinh 2sh_3\right), \qquad \alpha_{23} = \frac{\kappa h_3}{s\Delta}\sinh sh_3, \qquad \alpha_{24} = \frac{\kappa}{s\Delta}\left(h_3\cosh sh_3 - \frac{\kappa}{s}\sinh sh_3\right)$$

$$\alpha_{31} = -\frac{\kappa}{s\Delta}\sinh^2 sh_3, \qquad \alpha_{32} = -\frac{1}{\Delta}\left(h_3 - \frac{\kappa}{2s}\sinh 2sh_3\right), \quad \alpha_{33} = \frac{\kappa_3}{\Delta}\sinh sh_3,$$

$$\alpha_{34} = \frac{1}{\Delta}\left(h_3\cosh sh_3 - \frac{\kappa}{s}\sinh sh_3\right), \qquad \alpha_{41} = \frac{1}{\Delta}\left(h_3 + \frac{\kappa}{2s}\sinh 2sh_3\right), \qquad \alpha_{42} = -\frac{\kappa}{s\Delta}\sinh^2 sh_3$$

$$\alpha_{43} = -\frac{1}{\Delta}\left(\frac{\kappa}{s}\sinh sh_3 + h_3\cosh sh_3\right), \quad \alpha_{44} = -\frac{h_3}{\Delta}\sinh sh_3, \qquad \Delta(s) = h_3^2 - \frac{\kappa^2}{s^2}\sinh^2 sh_3$$